Lecture 6: Equivalent sets

Function: A function f from A to B is an assignment which assigns each element of A to a unique element of B. Equivalently, a function is a relation from A to B such that for each element $a \in A$ there is a unique element $b \in B$ such that $a\,R\,b$.
A and B are called the domain and co-domain of the function respectively. The set of all those elements in B which are mapped by some elements in A is called the range or image of f.

Types of Functions: Let $f : X \to Y$.

1. **Injective (one-one):** For every pair $x, Y \in X$, we have: if $f(x) = f(y)$, then $x = y$ Or $x \neq y$ implies $f(x) \neq f(y)$. Find the number of injective functions from from X to Y.
2. **Surjective (onto):** f is onto, if for each $b \in B$ there is $a \in A$ such that $f(a) = b$. Find the number of surjective functions from from X to Y.
3. **Bijective**: If it is one-one and onto. Find the number of bijective functions from X to Y.

Equivalent or similar sets: Sets X and Y are called equivalent (or similar), if there is a bijective map between them. In this case, we denote $X \approx Y$. We say that a set A is finite if $A = \varnothing$ or $A \approx I_n = \{1, 2, \ldots, n\}$.

Theorem: The relation $\approx$ is an equivalence relation in any collection of sets.

Examples:

1. Suppose $a, b \in \mathbb{R}$ and $a < b$. Then $(0, 1) \approx (a, b)$ by $f : (0, 1) \to (a, b)$ as $f(x) = a + (b - a)x$.
2. Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Then $(a, b) \approx (c, d)$.
3. $(0, 1] \approx [0, 1)$. Consider $f : (0, 1] \to [0, 1)$ as $f(x) = 1 - x$.
4. $\mathbb{R} \approx (0, \infty)$. Consider $f : \mathbb{R} \to (0, \infty)$ as $f(x) = e^x$.

Theorem Suppose $A \approx C$ and $B \approx D$. Then

1. $A \times B \approx C \times D$.
2. If $A \cap B = \varnothing$ and $C \cap D = \varnothing$, then $A \cup B \approx C \cup D$.

Proof 1: Since $A \approx C$ and $B \approx D$, there exist bijective functions $f : A \to C$ and $g : B \to D$. Define $h : A \times B \to C \times D$ as $h(a, b) = (f(a), g(b))$. Then h is bijective.

2: Exercise- Hint: Define $h : A \cup B \to C \cup D$ as

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Construction of bijective map: Let $f : X \to Y$ be a bijective map. Let $\{a_1, a_2, \ldots\} \subseteq X$. Then

1. If $c_1, c_2, \ldots, c_k$ are distinct objects not in X, then the function

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \setminus \{a_1, a_2, \ldots\} \\ f(a_i) & \text{if } x = c_i,\ i = 1, 2, \ldots, k \\ f(a_{i+k}) & \text{if } x = a_i,\ i \in \mathbb{N} \end{cases}$$

   is a bijection from $X \cup \{c_1, c_2, \ldots, c_k\}$ to Y.

2. If $c_1, c_2, \ldots$ are distinct objects not in X, then the function

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \setminus \{a_1, a_2, \ldots\} \\ f(a_{2n-1}) & \text{if } x = a_n,\ n \in \mathbb{N} \\ f(a_{2n}) & \text{if } x = c_n,\ n \in \mathbb{N} \end{cases}$$

   is a bijection from $X \cup \{c_1, c_2, \ldots\}$ to Y.

Examples: Let $A = [0, 1)$ and $B = (0, 2)$. Now we define bijective map between them as follows: Let $X = (0, 1)$ and $Y = (0, 2)$. Then $f : X \to Y$ defined as $f(x) = 2x$ is a bijective function. Let $\{a_1 = 1/2, a_2 = 1/3, a_3 = 1/4, \ldots\} \subseteq X$. Also let $\{c_1 = 0\}$. The the following function

$$h(x) = \begin{cases} f(x) & \text{if } x \in (0, 1) \setminus \{1/2, 1/3, \ldots\} \\ f(a_1 = 1/2) & \text{if } x = c_1 = 0 \\ f(a_{i+1}) & \text{if } x = a_i,\ i \in \mathbb{N} \end{cases}$$

is a bijection from $[0, 1)$ to $Y = (0, 2)$.

Lemma: Let $f : X \to Y$ be a function. Let $\{A_\alpha\}_{\alpha \in I}$ be a family of subsets of X. Then

$$f(\bigcup_{\alpha \in I} A_\alpha) = \bigcup_{\alpha \in I} f(A_\alpha).$$

Proof: See question in Tutorial sheet-2.

Lemma: Let $A, B \subseteq X$. If $f : X \to Y$ is one-one, then $f(A \setminus B) = f(A) \setminus f(B)$.
Proof: Let $x \in A \setminus B$. Then $f(x) \in f(A)$. To show that $f(x) \notin f(B)$. Suppose $f(x) \in f(B)$. Then $f(x) = f(b)$ for some $b \in B$. Since f is one-one, $x = b$, that is, $x \in B$. A contradiction. Conversely, let $y \in f(A) \setminus f(B)$. Then there exists $a \in A$ such that $f(a) = y$. To show that $a \notin B$. Suppose $a \in B$. Then $y = f(a) \in f(B)$. A contradiction.

Cantor-Schröder-Bernstein (CSB) Theorem: Let $f : X \to Y$ and $g : Y \to X$ is one-one. Then there exists a bijective function $h : X \to Y$, that is, $X \approx Y$.
Proof: if f is onto then f is the required map. So assume that f is not onto. Then $f(X) \subset Y$.

Let $B = Y \setminus f(X)$ and $\phi = f \circ g$. Let $A = B \cup \phi(B) \cup \phi^2(B) \cup \ldots = B \cup_{n=1}^{\infty} \phi^n(B)$.

Then $A \subseteq Y$ and $\phi(A) = \phi(B) \cup_{n=2}^{\infty} \phi^n(B) = \cup_{n=1}^{\infty} \phi^n(B)$.

Hence $A = B \cup \phi(A)$.

Note that $f(X) = Y \setminus B$ and $\phi(A) = f \circ g(A) = f(g(A)) \subseteq Y$.

Since f is one-one, $f(X \setminus g(A)) = f(X) \setminus f(g(A)) = (Y \setminus B) \setminus \phi(A) = Y \setminus (B \cup \phi(A)) = Y \setminus A$.

Thus the restriction of f to $X \setminus g(A)$ is a bijection onto $Y \setminus A$.

As g is one-one, $g : A \to g(A)$ is bijective, that is, $g^{-1} : g(A) \to A$ is bijective.

Then $h : X \to Y$ defined as

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \setminus g(A) \\ g^{-1}(x) & \text{if } x \in g(A) \end{cases}$$

is a bijection.

Examples:

1. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ by $f(n) = (n, 1)$ and $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $g(m, n) = 2^m 3^n$.
2. $(0, 1) \approx (0, 1]$. Define $f : (0, 1) \to (0, 1]$ as $f(x) = x$ and $g : (0, 1] \to (0, 1)$ as $g(x) = \frac{x}{2}$.